Existence of Solutions to Strong Set-Valued Generalized Vector Quasi-Equilibrium Problems

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Abstract. In his paper, using the fixed point theorem of KFG , studies the strong set-valued generalized vector quasi-equilibrium problem with domination structure in the conditions of convexity and continuity, and obtains the existence of solutions.

Introduction

The Set-Valued Generalized vector quasi-equilibrium Problem (SSGVQEP) includes as special cases of vector equilibrium problems (SVEP) [3-6]and the muti-objective generalized game problem.The SVEP includes as special cases the equilibrium problems,the vector inequality problems,the system of vector optimization problems and the muti-objective game problem .The muti-objective generalized game problem also includes the muti-objective game problem as a special case.

Motivated and inspired by research works mentioned above, in this paper, we consider a strong generalized vector quasi-equilibrium problem without assuming that the dual of the ordering cone has a weak∗compact base. It includes the strong vector equilibrium problem in and the generalized strong vector equilibrium problem as special cases. We establish an existence theorem by using Kakutani–Fan–Glicksberg fixed point theorem .

Preliminaries

Throughout this paper, let X,Y,Z be three real Hausdorff topological vector spaces, $\forall x\in D$, $C(x)$ be a closed convex pointed cone in Y and $\operatorname{int}C(x)\neq\varnothing$,Set-vaule mapping: $S:D\to 2^D$, $T:D\to 2^K$, $F:D\times K\times D\to 2^Z$.

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Set-Valued Generalized vector quasi-equilibrium（In short SGVQEP）:

Find $\bar{x}\in D$,such that $\bar{x}\in S(\bar{x})$, $\bar{y}\in T(\bar{x})$ and

$$F(\bar{x},\bar{y},u)\subset C(\bar{x}),\forall u\in S(\bar{x})$$

Definition 2.1 $T:X\to 2^Y$ is set-valued mapping.We say that:

(1) T is called upper semi-continuous at $x_0\in X$,for any neighborhood $U(T(x_0))$ of $T(x_0)$, there is a neighborhood $N(x_0)$ of x_0 such that $T(x)\subset U(T(x_0))$ for all $x\in N(x_0)$;

(2) T is called lower semi-continuous at $x_0\in X$,for all $y_0\in T(x_0)$ and any neighborhood $N(y_0)$ of y_0, there exists a neighborhood $N(x_0)$ of x_0 such that $\forall x\in N(x_0)$ ，有 $T(x)\cap N(y_0)\neq\varnothing$.

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(3) T is called continuous at $x_0 \in X$ if T is both upper semi-continuous and lower semi-continuous at x_0.

(4) T is upper-continuous and close set,then T is closed.

(5) T is closed ,if graph $Gr(T)=\{(x,y)\in X\times Y : y\in T(x)\}$ is a close set.

Definition 2.2 Let X,Y be two real Hausdorff topological vector spaces, $C: X \to 2^Y$ is a set-vauled mapping, $\forall x\in X$, $C(x)$ is a closed convex pointed cone .Called G is $C(x)$ quasi-convex, $\forall x_i^1, x_i^2 \in X_i$, $\forall \lambda \in (0,1)$,we have either

$$G_i(\lambda x_i^1+(1-\lambda)x_i^2)\subset G_i(x_i^1)+C_i(x)$$

or

$$G_i(\lambda x_i^1+(1-\lambda)x_i^2)\subset G_i(x_i^2)+C_i(x).$$

Lemma2.1（KFG fixed point theorem）Let X,Y be two real Hausdorff topological vector spaces, $A\subset X$ and A is compactness. $G: X\to 2^Y$ is upper-continuous,and $\forall x\in A, G(x)$ is a closed convex set,then G have fixed point in A .

The Existence of Sgvqep

Theorem 3.1 let X,Y,Z be three real Hausdorff topological vector spaces, , $D\subset X$, $K\subset Y$ is compactness and convex set.assume that:

（i） $S(\bullet)$ is upper-continuous,close and convex on D ;

（ii） $T(\bullet)$ is upper-continuous,compactness and convex on D ;

（iii） $F(\bullet,\bullet,\bullet)$ is lower-continuous、 C upper-continuous on $D\times K\times D$ $\forall (y,u)\in K\times D$, and $F(\bullet,y,u)$ is properly quasi-convex on D .

Then the SGVQEP has a solution.

Proof：Using KFG fixed point theorem,The proof is divided into four parts：

（I）Define $\forall (x,y)\in D\times K$, $A: D\times K\to 2^D$ as follows：

$$A(x,y)=\{v\in S(x): F(v,y,u)\subset C(x), \forall u\in S(x)\}$$

First,we will prove $A(x,y)$ is close set .In fact ,let $\{v_\alpha\}\subset A(x,y)$ and $v_\alpha \to v_0$,we will prove $v_0\in A(x,y)$.

Since $\{v_\alpha\}\subset A(x,y)$,we have

$$v_\alpha \in S(x) \qquad (1)$$

$$F(v_\alpha, y,u)\subset C(x), \forall u\in S(x) \qquad (2)$$

By（1） and $S(x)$ is close,then $v_0\in S(x)$.

Thus,we will prove $\forall u\in S(x)$, $F(v_0,y,u)\subset C(x)$ .In fact ,there exists $u_0\in S(x)$,such that $F(v_0,y,u_0)\subset C(x)$ .Then there exists $z_0\in F(v_0,y,u_0)$,such that $z_0\notin C(x)$ .Since F is lower-continuous ,there exists $z_\alpha \in F(v_\alpha,y,u_\alpha)$,such that $z_\alpha\to z_0$ .by $z_0\notin C(x)$,exist α_0 ,when $\forall \alpha\ge\alpha_0$,we have

$$z_\alpha \notin C(x) \quad (3)$$

Which contradicts to（2）.Thus we get prove $A(x,y)$ is close set .

（II）We will prove $\forall (x,y) \in D \times K$, $A(x,y)$ is convex set.

$\forall v_1, v_2 \in A(x,y)$, $\forall t \in [0,1]$,let $v_t = tv_1 + (1-t)v_2$,we will prove $v_t \in A(x,y)$.

By $v_1, v_2 \in A(x,y)$,we have

$$v_i \in S(x) \quad (4)$$

$$F(v_i, y, u) \subset C(x), \forall u \in S(x) \quad (5)$$

（其中 $i = 1,2$ ）.By(4)and $S(x)$ is convex set,we get $v_t \in S(x)$.

Next,we will prove: $\forall u \in S(x)$ $F(v_t, y, u) \subset C(x)$.

In fact,if there exists $u_t \in S(x)$ such that $F(v_t, y, u_t) \not\subset C(x)$.

since F is $C(x)$ quasi-convex,then we have either $F(v_1, y, u) \subset F(v_t, y, u) + C(x)$ or $F(v_2, y, u) \subset F(v_t, y, u) + C(x)$ .without loss of generality ,we can assume that $F(v_1, y, u) \subset F(v_t, y, u) + C(x)$ .By(5)we get

$$F(v_1, y, u) \subset F(v_t, y, u) + C(x) \not\subset C(x) + C(x) \subset C(x)$$

Which contradicts to $F(v_1, y, u) \subset F(v_t, y, u) + C(x)$ .Thus $A(x,y)$ is convex set.

（III）We will prove A is upper semi-continuous .

In fact ,since $S(x)$ is closed, D is compactness and $S(x) \subset D$,then $S(x)$ is compactness.

$\forall (x,y) \in D \times K$, $A(x,y) \subset S(x)$,by Lemma 2.2we should prove A is closed mapping.

Let $\{(x_\alpha, y_\alpha, v_\alpha)\} \subset Gr(A)$, $(x_\alpha, y_\alpha, v_\alpha) \to (x_0, y_0, v_0)$,we will prove $v_0 \in A(x_0, y_0)$ .i.e. $v_0 \in S(x_0)$ and $F(v_0, y_0, u) \subset C(x_0)$， $\forall u \in S(x_0)$.

since $\{(x_\alpha, y_\alpha, v_\alpha)\} \subset Gr(A)$,we get $v_\alpha \subset A(x_\alpha, y_\alpha)$ and

$$v_\alpha \in S(x_\alpha) \quad (7)$$

$$F(v_\alpha, y_\alpha, u) \subset C(x_\alpha), \forall u \in S(x_\alpha) \quad (8)$$

$\forall x \in D$, $S(x)$ is close set and S is upper semi-continuous .then S is closed mapping.by（7）we have $v_0 \in S(x_0)$ .next we will prove $F(v_0, y_0, u) \subset C(x_0)$， $\forall u \in S(x_0)$.

If exist $u_0 \in S(x_0)$,we have $F(v_0, y_0, u_0) \not\subset C(x_0)$ .then ,there exists open neighbourhood U ,such that

$$F(v_0, y_0, u_0) + U \not\subset C(x_0) \quad (9)$$

Since F is C upper-continuous ,then for this open neighbourhood U ,we have

$F(v_0', y_0', u_0') \subset F(v_0, y_0, u_0) + U + C(x_0)$, $\forall (v_0', y_0', u_0') \in (v_0, y_0, u_0) + U$

Then ,there exist α_0 ,for $\forall \alpha \geq \alpha_0$,we get $(v_\alpha, y_\alpha, u_\alpha) \in (v_0, y_0, u_0) + U$,then by(9),we have

$F(v_\alpha, y_\alpha, u_\alpha) \subset F(v_0, y_0, u_0) + U + C(x_0) \not\subset C(x_0) + C(x_0) \subset C(x_0)$

Which contradicts to (8). Thus A is upper semi-continuous .

（IV）Definition $M: D\times K \to 2^{D\times K}$： $\forall (x,y)\in D\times K$, $M(x,y)=(A(x,y),T(x))$ .since A,T are upper semi-continuous with compact values and convex,then M is also are upper semi-continuous with compact values and convex. By KFG fixed point theorem ,there exists $(\bar{x},\bar{y})\in M(\bar{x},\bar{y})$ such that $(\bar{x},\bar{y})\in (A(\bar{x},\bar{y}),T(\bar{x}))$ .which implies that

find $\bar{x}\in S(\bar{x})$, $\bar{y}\in T(\bar{x})$ such that $F(\bar{x},\bar{y},u)\subset C(\bar{x}),\forall u\in S(\bar{x})$.

Thus SSGVQEP have solution.

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